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# Step refinable functions and orthogonal MRA on $p$-adic Vilenkin groups [1]

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**Abstract.** We find the necessary and sufficient conditions for refinable step function under which this function generates an orthogonal MRA in the $L_2(\mathfrak{G})$ -spaces on Vilenkin groups $\mathfrak{G}$. We consider a class of refinable step functions for which the mask $m_0(\chi)$ is constant on cosets $\mathfrak{G}_{-1}^{\perp}$ and its modulus $|m_0(\chi)|$ takes two values only: 0 and 1. We will prove that any refinable step function $\varphi$ from this class that generates an orthogonal MRA on $p$-adic Vilenkin group $\mathfrak{G}$ has Fourier transform with condition $\operatorname{supp}\hat{\varphi}(\chi) \subset \mathfrak{G}_{p-2}^{\perp}$. We show the sharpness of this result too.

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## Introduction

Foundations for wavelet analysis theory on locally compact groups have been lay in the monograph [1]. In articles [2-4] first examples of orthogonal wavelets on the dyadic Cantor group are constructed and their properties are studied. The general scheme for the construction of wavelets is based on the notion of multiresolution analysis (MRA in the sequel) introduced by Y. Meyer and S. Mallat [5, 6]. Yu.Farkov [7-12] found necessary and sufficient conditions for a refinable function under which this function generates an orthogonal MRA in the $L_2(\mathfrak{G})$ -spaces on the Vilenkin group $\mathfrak{G}$. These conditions use the Strang-Fix and the modified Cohen properties. In [10] this construction to the p = 3 case in a concrete fashion are given. In [11], some algorithms for constructing orthogonal and biorthogonal compactly supported wavelets

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on Vilenkin groups are suggested. In [7-11] two types of orthogonal wavelets examples are constructed: step functions and sums of Vilenkin series.

In these examples all step refinable functions have a support $\operatorname{supp}\hat{\varphi}(\chi) \subset \mathfrak{G}_1^\perp = \mathfrak{G}_0^\perp \mathcal{A}$ where $\mathfrak{G}_0^\perp$ is the unit ball in the character group and $\mathcal{A}$ is a dilation operator. Therefore there is an assumption that a step refinable function which generates an orthogonal MRA on Vilenkin group $\mathfrak{G}$ has a Fourier transform with support $\operatorname{supp}\hat{\varphi}(\chi) \subset \mathfrak{G}_1^\perp$ . We will prove that it is not true. We consider a class of refinable step functions for which the mask $m_0(\chi)$ is constant on cosets $\mathfrak{G}_{-1}^\perp$ and its modulus $|m_0(\chi)|$ takes two values only: 0 and 1. We will prove that any refinable step function $\varphi$ from this class that generates an orthogonal MRA on $p$-adic Vilenkin group $\mathfrak{G}$ has Fourier transform with condition $\operatorname{supp}\hat{\varphi}(\chi) \subset \mathfrak{G}_{p-2}^\perp$. We show the sharpness of this result too.

We should note that in the p-adic analysis, the situation is different. S. Albeverio, S. Evdokimov, M. Skopina [12] proved, that if a refinable step function $\varphi$ generates an orthogonal $p$-adic MRA, then $\hat{\varphi}(\chi) \subset \mathfrak{G}_0^\perp$.

# 1 Preliminaries

We will consider the Velenkin group as a locally compact zero-dimensional Abelian group with additional condition $p_n g_n = 0$. Therefore we start with some basic notions and facts related to analysis on zero-dimensional groups. A topological group in which the connected component of 0 is 0 is usually referred to as a *zero-dimensional group*. If a separable locally compact group $(G, \dot{+})$ is zero-dimensional, then the topology on it can be generated by means of a descending sequence of subgroups.

The converse statement holds for all topological groups (see [13, Ch. 1, § 3]). So, for a locally compact group, we are going to say 'zero-dimensional group' instead of saying 'a group with topology generated by a sequence subgroups'.

Let $(G, \dot{+})$ be a locally compact zero-dimensional Abelian group with the topology generated by a countable system of open subgroups

$$\cdots \supset G_{-n} \supset \cdots \supset G_{-1} \supset G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots$$

where

$$\bigcup_{n=-\infty}^{+\infty} G_n = G, \qquad \bigcap_{n=-\infty}^{+\infty} G_n = \{0\}$$

(0 is the null element in the group $G$). Given any fixed $N \in \mathbb{Z}$, the subgroup $G_N$ is a compact Abelian group with respect to the same operation $\dot{+}$

under the topology generated by the system of subgroups

$$G_N \supset G_{N+1} \supset \cdots \supset G_n \supset \cdots .$$

As each subgroup $G_n$ is compact, it follows that each quotient group $G_n/G_{n+1}$ is finite (say, of order $p_n$). We may always assume that all $p_n$ are prime numbers. We will name such chain as *basic chain.* In this case, a base of the topology is formed by all possible cosets $G_n\dot{+}g$, $g \in G$.

We further define the numbers $(\mathfrak{m}_n)_{n=-\infty}^{+\infty}$ as follows:

$$\mathfrak{m}_0 = 1, \qquad \mathfrak{m}_{n+1} = \mathfrak{m}_n \cdot p_n.$$

Clearly, for $n \geq 1$,

$$\mathfrak{m}_n = p_0 p_1 \cdots p_{n-1}, \qquad \mathfrak{m}_{-n} = \frac{1}{p_{-1}p_{-2}\cdots p_{-n}}.$$

The collection of all such cosets $G_n\dot{+}g$, $n \in \mathbb{Z}$, along with the empty set form the semiring $\mathscr{K}$. On each coset $G_n\dot{+}g$ we define the measure $\mu$ by $\mu(G_n\dot{+}g) = \mu G_n = 1/m_n$. So, if $n \in \mathbb{Z}$ and $p_n = p$, we have $\mu G_n \cdot \mu G_{-n} = 1$. The measure $\mu$ can be extended from the semiring $\mathscr{K}$ onto the $\sigma$-algebra (for example, by using Caratheodory's extension). This gives the translation invariant measure $\mu$, which agrees on the Borel sets with the Haar measure on $G$. Further, let $\displaystyle\int_G f(x)\,d\mu(x)$ be the absolutely convergent integral of the measure $\mu$.

Given an $n \in \mathbb{Z}$, take an element $g_n \in G_n \setminus G_{n+1}$ and fix it. Then any $x \in G$ has a unique representation of the form

$$x = \sum_{n=-\infty}^{+\infty} a_n g_n, \qquad a_n = \overline{0, p_n - 1}. \tag{1.1}$$

The sum (1.1) contain finite number of terms with negative subscripts, that is,

$$x = \sum_{n=m}^{+\infty} a_n g_n, \qquad a_n = \overline{0, p_n - 1}, \quad a_m \neq 0. \tag{1.2}$$

We will name system $(g_n)_{n\in\mathbb{Z}}$ as *a basic system.*

Classical examples of zero-dimensional groups are Vilenkin groups and groups of $p$-adic numbers (see [13, Ch. 1, § 2]). A direct sum of cyclic groups $Z(p_k)$ of order $p_k$, $k \in \mathbb{Z}$, is called a *Vilenkin group.* This means that the elements of a Vilenkin group are infinite sequences $x = (x_k)_{k=-\infty}^{+\infty}$ such that:

1) $x_k = \overline{0, p_k - 1}$;

2) only a finite number of $x_k$ with negative subscripts are different from zero;

3) the group operation $\dot{+}$ is the coordinate-wise addition modulo $p_k$, that is,

$$x\dot{+}y = (x_k\dot{+}y_k), \qquad x_k\dot{+}y_k = (x_k + y_k) \mod p_k.$$

A topology on such group is generated by the chain of subgroups

$$G_n = \{x \in G : x = (\dots, 0, 0, \dots, 0, x_n, x_{n+1}, \dots),\ x_\nu = \overline{0, p_\nu - 1},\ \nu \geq n\}.$$

The elements $g_n = (\dots, 0, 0, 1, 0, 0, \dots)$ form a basic system. From definition of the operation $\dot{+}$ we have $p_n g_n = 0$. Therefore we will name a zero-dimensional group $(G, \dot{+})$ with the condition $p_n g_n = 0$ as Vilenkin group.

The group $\mathbb{Q}_p$ of all $p$-adic numbers ($p$ is a prime number) also consists of sequences $x = (x_k)_{k=-\infty}^{+\infty}$, $x_k = \overline{0, p-1}$, only a finite number of $x_k$ with negative subscripts being different from zero. However, the group operation in $\mathbb{Q}_p$ is defined differently. Namely, given elements

$$x = (\dots, 0, \dots, 0, x_N, x_{N+1}, \dots) \text{ and } y = (\dots, 0, \dots, 0, y_N, y_{N+1}, \dots) \in \mathbb{Q}_p,$$

we again add them coordinate-wise, but whereas in a Vilenkin group $x_n\dot{+}y_n = (x_n + y_n) \mod p$ (that is, a 1 is not carried to the next $(n+1)$th position), the corresponding $p$-adic summation has the property that the 1 occuring as a result of the addition of $x_n + y_n$ is carried to the next $(n+1)$th position. We endow the group $\mathbb{Q}_p$ with the topology generated by the same system of subgroups $G_n$ as for a Vilenkin group. Similarly, as a $(g_n)$, we may again take the same sequence.

By $X$ denote the collection of the characters of a group $(G, \dot{+})$; it is a group with respect to multiplication too. Also let $G_n^\perp = \{\chi \in X : \forall\, x \in G_n\ , \chi(x) = 1\}$ be the annihilator of the group $G_n$. Each annihilator $G_n^\perp$ is a group with respect to multiplication, and the subgroups $G_n^\perp$ form an increasing sequence

$$\dots \subset G_{-n}^\perp \subset \dots \subset G_0^\perp \subset G_1^\perp \subset \dots \subset G_n^\perp \subset \dots \tag{1.3}$$

with

$$\bigcup_{n=-\infty}^{+\infty} G_n^\perp = X \quad \text{and} \quad \bigcap_{n=-\infty}^{+\infty} G_n^\perp = \{1\},$$

the quotient group $G_{n+1}^\perp / G_n^\perp$ having order $p_n$. The group of characters $X$ may be equipped with the topology using the chain of subgroups (1.3), the family of the cosets $G_n^\perp \cdot \chi$, $\chi \in X$, being taken as a base of the topology. The collection of such cosets, along with the empty set, forms the semiring $\mathscr{X}$.

Given a coset $G_n^\perp \cdot \chi$, we define a measure $\nu$ on it by $\nu(G_n^\perp \cdot \chi) = \nu(G_n^\perp) = m_n$ (so that always $\mu(G_n)\nu(G_n^\perp) = 1$). The measure $\nu$ can be extended onto the $\sigma$-algebra of measurable sets in the standard way. One then forms the absolutely convergent integral $\int_X F(\chi)\, d\nu(\chi)$ of this measure.

The value $\chi(g)$ of the character $\chi$ at an element $g \in G$ will be denoted by $(\chi, g)$. The Fourier transform $\widehat{f}$ of an $f \in L_2(G)$ is defined as follows

$$\widehat{f}(\chi) = \int_G f(x)\overline{(\chi, x)}\, d\mu(x) = \lim_{n\to+\infty} \int_{G_{-n}} f(x)\overline{(\chi, x)}\, d\mu(x),$$

the limit being in the norm of $L_2(X)$. For any $f \in L_2(G)$, the inversion formula is valid

$$f(x) = \int_X \widehat{f}(\chi)(\chi, x)\, d\nu(\chi) = \lim_{n\to+\infty} \int_{G_n^\perp} \widehat{f}(\chi)(\chi, x)\, d\nu(\chi);$$

here the limit also signifies the convergence in the norm of $L_2(G)$. If $f, g \in L_2(G)$ then the Plancherel formula is valid

$$\int_G f(x)\overline{g(x)}\, d\mu(x) = \int_X \widehat{f}(\chi)\overline{\widehat{g}(\chi)}\, d\nu(\chi).$$

Endowed with this topology, the group of characters $X$ is a zero-dimensional locally compact group; there is, however, a dual situation: every element $x \in G$ is a character of the group $X$, and $G_n$ is the annihilator of the group $G_n^\perp$.

The union of disjoint sets $E_j$ we will denote by $\bigsqcup E_j$.

## 2 Rademacher functions and dilation operator

In this section we will consider zero-dimensional groups with condition $p_n = p$ for any $n \in \mathbb{Z}$. In this case we define the mapping $\mathcal{A}\colon G \to G$ by $\mathcal{A}x := \sum_{n=-\infty}^{+\infty} a_n g_{n-1}$, where $x = \sum_{n=-\infty}^{+\infty} a_n g_n \in G$. The mapping $\mathcal{A}$ is called a dilation operator if $\mathcal{A}(x\dot{+}y) = \mathcal{A}x\dot{+}\mathcal{A}y$ for all $x, y \in G$. By definition, put $(\chi\mathcal{A}, x) = (\chi, \mathcal{A}x)$. A character $r_n \in G_{n+1}^\perp \setminus G_n^\perp$ is called the Rademacher function. Let us denote

$$H_0 = \{h \in G : h = a_{-1}g_{-1}\dot{+}a_{-2}g_{-2}\dot{+}\dots\dot{+}a_{-s}g_{-s}, s \in \mathbb{N}\},$$

$$H_0^{(s)} = \{h \in G : h = a_{-1}g_{-1}\dot{+}a_{-2}g_{-2}\dot{+}\dots\dot{+}a_{-s}g_{-s}\}, s \in \mathbb{N}.$$

The set $H_0$ is an analog of the set $\mathbb{N}$.

**Lemma 2.1** *For any zero-dimensional group*
*1)* $\int\limits_{G_0^\perp}(\chi,x)\,d\nu(\chi)=\mathbf{1}_{G_0}(x)$, *2)* $\int\limits_{G_0}(\chi,x)\,d\mu(x)=\mathbf{1}_{G_0^\perp}(\chi)$.

The first equation it was proved in [14], the second equation is dual to first.

**Lemma 2.2** *If* $p_n=p$ *for any* $n\in\mathbb{Z}$ *and the mapping* $\mathcal{A}$ *is additive then*
*1)* $\int\limits_{G_n^\perp}(\chi,x)\,d\nu(\chi)=p^n\mathbf{1}_{G_n}(x)$,
*2)* $\int\limits_{G_n}(\chi,x)\,d\mu(x)=\frac{1}{p^n}\mathbf{1}_{G_n^\perp}(\chi)$.

**Proof.** First we prove the equation 1). Using equations

$$\int\limits_X f(\chi\mathcal{A})\,d\nu(\chi)=p\int\limits_X f(\chi)\,d\nu(\chi),\quad \mathbf{1}_{G_n^\perp}(x)=\mathbf{1}_{G_0}(\mathcal{A}^n x),$$

and Lemma 2.1 we have

$$\int\limits_{G_n^\perp}(\chi,x)\,d\nu(\chi)=\int\limits_X \mathbf{1}_{G_n^\perp}(\chi)(\chi,x)\,d\nu(\chi)=p^n\int\limits_X(\chi\mathcal{A}^n,x)\mathbf{1}_{G_n^\perp}(\chi\mathcal{A}^n)\,d\nu(\chi)=$$

$$=p^n\int\limits_X(\chi,\mathcal{A}^n x)\mathbf{1}_{G_0^\perp}(\chi)\,d\nu(\chi)=p^n\mathbf{1}_{G_0}(\mathcal{A}^n x)=p^n\mathbf{1}_{G_n}(x).$$

The second equation is proved by analogy. $\square$

**Lemma 2.3** *Let* $\chi_{n,s}=r_n^{\alpha_n}r_{n+1}^{\alpha_{n+1}}\dots r_{n+s}^{\alpha_{n+s}}$ *be a character does not belong to* $G_n^\perp$. *Then*

$$\int\limits_{G_n^\perp\chi_{n,s}}(\chi,x)\,d\nu(\chi)=p^n(\chi_{n,s},x)\mathbf{1}_{G_n}(x).$$

**Proof.** By analogy with previously we have

$$\int\limits_{G_n^\perp\chi_{n,s}}(\chi,x)\,d\nu(\chi)=\int\limits_X \mathbf{1}_{G_n^\perp}(\chi)(\chi_{n,s}\chi,x)\,d\nu(\chi)=$$

$$\int\limits_{G_n^\perp}(\chi_{n,s},x)(\chi,x)\,d\nu(\chi)=p^n(\chi_{n,s},x)\mathbf{1}_{G_n}(x).\ \square$$

**Lemma 2.4** *Let $h_{n,s} = a_{n-1}g_{n-1}\dot{+}a_{n-2}g_{n-2}\dot{+}\ldots\dot{+}a_{n-s}g_{n-s} \notin G_n$. Then*

$$\int\limits_{G_n\dot{+}h_{n,s}} (\chi, x)\, d\mu(x) = \frac{1}{p^n}(\chi, h_{n,s})\mathbf{1}_{G_n^\perp}(\chi).$$

This lemma is dual to lemma 2.3.

**Definition 2.1** *Let $M, N \in \mathbb{N}$. Denote by $\mathfrak{D}_M(G_{-N})$ the set of step-functions $f \in L_2(G)$ such that 1)*$\operatorname{supp} f \subset G_{-N}$*, and 2) $f$ is constant on cosets $G_M$. Similarly is defined $\mathfrak{D}_{-N}(G_M^\perp)$.*

**Lemma 2.5** *Let $M, N \in \mathbb{N}$. $f \in \mathfrak{D}_M(G_{-N})$ if and only if $\hat{f} \in \mathfrak{D}_{-N}(G_M^\perp)$.*

**Proof.** 1) Let $f$ be a constant on cosets $G_M\dot{+}g$ and $\operatorname{supp} f \subset G_{-N}$. Let us show that $\operatorname{supp} \hat{f} \subset G_M^\perp$. Let $\chi \notin G_M^\perp$. Then

$$\hat{f}(\chi) = \int\limits_G f(x)\overline{(\chi, x)}\, d\mu(x) = \int\limits_{G_{-N}} f(x)\overline{(\chi, x)}\, d\mu(x) =$$

$$= \sum_{h_{M,N}\in H_M^N} \int\limits_{G_M\dot{+}h_{M,N}} f(x)\overline{(\chi, x)}\, d\mu(x),$$

where

$$H_M^N = \{h_{M,N} = a_{M-1}g_{M-1}\dot{+}a_{M-2}g_{M-2}\dot{+}\ldots\dot{+}a_{-N}g_{-N}\}.$$

By lemma 2.4

$$\hat{f}(\chi) = \sum f(G_M\dot{+}h_{M,N}) \int\limits_{G_M\dot{+}h_{M,N}} \overline{(\chi, x)}\, d\mu(x) =$$

$$= \sum f(G_M\dot{+}h_{M,N})\frac{1}{p^M}\overline{(\chi, h_{M,N})}\mathbf{1}_{G_M^\perp}(\chi) = 0.$$

Now we will show that $\hat{f}$ is constant on cosets $G_{-N}^\perp\zeta$. Indeed let $\chi \in G_{-N}^\perp\zeta$ and $\zeta = r_{-N}^{\alpha_{-N}}r_{-N+1}^{\alpha_{-N+1}}\ldots r_{-N+s}^{\alpha_{-N+s}}$. Then $\chi = \chi_{-N}\zeta$ where $\chi_{-N} \in G_{-N}^\perp$. Therefore

$$\hat{f}(\chi) = \int\limits_{G_{-N}} f(x)\overline{(\chi, x)}\, d\mu(x) = \int\limits_{G_{-N}} f(x)\overline{(\chi_{-N}\zeta, x)}\, d\mu(x) = \int\limits_{G_{-N}} f(x)\overline{(\zeta, x)}\, d\mu(x).$$

It means that $\hat{f}(\chi)$ depends only on $\zeta$. The first part is proved. The second part is proved similarly. $\square$

**Lemma 2.6** *Let $\varphi \in L_2(G)$. The system $(\varphi(x\dot{-}h))_{h\in H_0}$ is orthonormal if and only if the system $\left(p^{\frac{n}{2}}\varphi(\mathcal{A}^n x\dot{-}h)\right)_{h\in H_0}$ is orthonormal.*

**Proof**. This lemma follows from the equation

$$\int\limits_G p^{\frac{n}{2}}\varphi(\mathcal{A}^n x\dot{-}h)p^{\frac{n}{2}}\overline{\varphi(\mathcal{A}^n x\dot{-}g)}\,d\mu = \int\limits_G \varphi(x\dot{-}h)\overline{\varphi(x\dot{-}g)}\,d\mu. \ \square$$

## 3 MRA on Vilenkin groups

In what follows we will consider groups $G$ for which $p_n = p$ and $pg_n = 0$ for any $n \in \mathbb{Z}$. We now that it is a Vilenkin group. We will denote a Vilenkin group as $\mathfrak{G}$. In this group we can chouse Rademacher functions in various ways. We define Rademacher functions by the equation

$$\left(r_n, \sum_{k\in\mathbb{Z}} a_k g_k\right) = \exp\left(\frac{2\pi i}{p}a_n\right).$$

In this case

$$(r_n, g_k) = \exp\left(\frac{2\pi i}{p}\delta_{nk}\right).$$

Our main objective is to find a refinable step-function that generates an orthogonal MRA on Vilenkin group.

**Definition 3.1** *A family of closed subspaces $V_n$, $n \in \mathbb{Z}$, is said to be a multiresolution analysis of $L_2(\mathfrak{G})$ if the following axioms are satisfied:*

*A1)* $V_n \subset V_{n+1}$;

*A2)* $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L_2(\mathfrak{G})$ *and* $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$;

*A3)* $f(x) \in V_n \iff f(\mathcal{A}x) \in V_{n+1}$ *($\mathcal{A}$ is a dilation operator);*

*A4)* $f(x) \in V_0 \implies f(x\dot{-}h) \in V_0$ *for all $h \in H_0$; ($H_0$ is analog of $\mathbb{Z}$).*

*A5) there exists a function $\varphi \in L_2(\mathfrak{G})$ such that the system $(\varphi(x\dot{-}h))_{h\in H_0}$ is an orthonormal basis for $V_0$.*

*A function $\varphi$ occurring in axiom A5 is called a scaling function.*

Next we will follow the conventional approach. Let $\varphi(x) \in L_2(\mathfrak{G})$, and suppose that $(\varphi(x\dot{-}h))_{h\in H_0}$ is an orthonormal system in $L_2(\mathfrak{G})$. With the function $\varphi$ and the dilation operator $\mathcal{A}$, we define the linear subspaces $L_j =$

$(\varphi(\mathcal{A}^j x \dot{-} h))_{h\in H_0}$ and closed subspaces $V_j = \overline{L_j}$. It is evident that the functions $p^{\frac{n}{2}}\varphi(\mathcal{A}x \dot{-} h)_{h\in H_0}$ form an orthonormal basis for $V_n$, $n \in \mathbb{Z}$. Therefore the axiom A4 is fulfilled. If subspaces $V_j$ form a MRA, then the function $\varphi$ is said to *generate* an MRA in $L_2(\mathfrak{G})$. If a function $\varphi$ generates an MRA, then we obtain from the axiom A1

$$\varphi(x) = \sum_{h\in H_0} \beta_h \varphi(\mathcal{A}x \dot{-} h) \quad \left(\sum |\beta_h|^2 < +\infty\right). \tag{3.1}$$

Therefore we will look up a function $\varphi \in L_2(\mathfrak{G})$, which generates an MRA in $L_2(\mathfrak{G})$, as a solution of the refinement equation (3.1), A solution of refinement equation (3.1) is called a *refinable function.*

**Lemma 3.1** *Let $\varphi \in \mathfrak{D}_M(\mathfrak{G}_{-N})$ be a solution of (3.1). Then*

$$\varphi(x) = \sum_{h\in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} h) \tag{3.2}$$

**Proof.** Let us write $\varphi(x)$ in the form

$$\varphi(x) = \sum_{h\in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} h) + \sum_{h\notin H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} h). \tag{3.3}$$

If $x \in \mathfrak{G}_{-N}$, then $\mathcal{A}x \in \mathfrak{G}_{-N-1}$. Therefore $\mathcal{A}x = b_{-N-1}g_{-N-1} \dot{+} b_{-N}g_{-N} \dot{+} \ldots$. If $h \notin H_0^{(N+1)}$, then

$$h = a_{-1}g_{-1} \dot{+} \ldots \dot{+} a_{-N-1}g_{-N-1} \dot{+} a_{-N-2}g_{-N-2} \dot{+} \ldots \dot{+} a_{-N-s}g_{-N-s},$$

and $a_{-N-2}g_{-N-2} \dot{+} \ldots \dot{+} a_{-N-s}g_{-N-s} \neq 0$. Hence $\mathcal{A}x \dot{-} h \notin H_0^{(N+1)}$ and $\varphi(\mathcal{A}x \dot{-} h) = 0$. This means that

$$\sum_{h\notin H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} h) = 0$$

when $x \in \mathfrak{G}_{-N}$.

Let $x \notin \mathfrak{G}_{-N}$. Then $\varphi(x) = 0$ and $\mathcal{A}x \notin \mathfrak{G}_{-N-1}$. Hence

$$\mathcal{A}x = \sum_{k=-N-s}^{-N-2} b_k g_k \dot{+} \sum_{k=-N-1}^{+\infty} b_k g_k.$$

If $h \in H_0^{(N+1)}$, then $h = a_{-1}g_{-1} \dot{+} \ldots \dot{+} a_{-N}g_{-N} \dot{+} a_{-N-1}g_{-N-1}$, and consequently $\mathcal{A}x \dot{-} h \notin \mathfrak{G}_{-N-1}$. Therefore

$$\sum_{h\in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} h) = 0.$$

Using equation (3.3) we obtain finally

$$\sum_{h \notin H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} h) = 0,$$

and lemma is proved. $\square$

**Theorem 3.2** *Let $\varphi \in \mathfrak{D}_M(\mathfrak{G}_{-N})$ and let $(\varphi(x \dot{-} h))_{h \in H_0}$ be an orthonormal system. $V_n \subset V_{n+1}$ if and only if the function $\varphi(x)$ is a solution of refinement equation (3.2).*

**Proof.** First we prove that $V_n \subset V_{n+1}$ if and only if $V_0 \subset V_1$. Indeed, let $V_0 \subset V_1$ and $f \in V_n$. Then

$$f(x) = \sum_h c_h \varphi(\mathcal{A}^n x \dot{-} h) \Rightarrow f(\mathcal{A}^{-n} x) = \sum_h c_h \varphi(x \dot{-} h) \Rightarrow f(\mathcal{A}^{-n} x) \in V_0 \Rightarrow$$

$$\Rightarrow f(\mathcal{A}^{-n} x) \in V_1 \Rightarrow f(\mathcal{A}^{-n} x) = \sum_h \gamma_h \varphi(\mathcal{A}x \dot{-} h) \Rightarrow$$

$$\Rightarrow f(x) = \sum_h \gamma_h \varphi(\mathcal{A}^{n+1} x \dot{-} h) \Rightarrow f \in V_{n+1}.$$

So we have, $V_n \subset V_{n+1}$. The converse is proved by analogy.

Now we prove that $V_0 \subset V_1$ if and only if the function $\varphi(x)$ is a solution of the refinement equation (3.2). The necessity is evident. Let $\varphi$ be a solution of (3.2). We take $f \in \operatorname{span}(\varphi(x \dot{-} h))_{h \in H_0}$. Then

$$f(x) = \sum_{\tilde{h} \in H_0^{(m)}} c_{\tilde{h}} \varphi(x \dot{-} \tilde{h})$$

for some $m \in \mathbb{N}$.

Since $\varphi$ is a solution of (3.2) then we can write $f$ in the form

$$f(x) = \sum_{\tilde{h} \in H_0^{(m)}} c_{\tilde{h}} \sum_{h \in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x \dot{-} (\mathcal{A}\tilde{h} \dot{+} h)).$$

Since $\tilde{h} \in H_0^{(m)}$ then $\mathcal{A}\tilde{h} \in H_0$. Therefore $\mathcal{A}\tilde{h} \dot{+} h \in H_0$. This means that $f \in \operatorname{span}(\varphi(\mathcal{A}x \dot{-} h))_{h \in H_0}$. It follows $V_0 \subset V_1$. $\square$

**Theorem 3.3** *Let $(\varphi(x \dot{-} h))_{h \in H_0}$ be an orthonormal basis in $V_0$. Then $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$.*

**Proof.** Let $f \in V_{-n}$ for some $n \in \mathbb{N}$. Then $f(\mathcal{A}^n x) \in V_0$. Since the system $(\varphi(x \dot{-} h))_{h\in H_0}$ is orthonormal we have the equality

$$\frac{1}{p^n}\sum_{h\in H_0}\left|\int\limits_{\mathfrak{G}} f(x)\varphi(\mathcal{A}^{-n}x\dot{-}h)\,d\mu\right|^2 = \sum_{h\in H_0}\left|\int\limits_{\mathfrak{G}} f(\mathcal{A}^n x)\varphi(x\dot{-}h)\,d\mu\right|^2 =$$

$$= ||f(\mathcal{A}^n x)||_2^2 = \int\limits_{\mathfrak{G}} |f(\mathcal{A}^n x)|^2\,d\mu = \frac{1}{p^n}||f||_2^2.$$

It is evident that $(p^{\frac{n}{2}}\varphi(\mathcal{A}^n x\dot{-}h))_{h\in H_0}$ is orthonormal basis in $V_n$. Therefore

$$||f||_2^2 = p^n\sum_{h\in H_0}\left|\int\limits_{\mathfrak{G}} f(x)\varphi(\mathcal{A}^{-n}x\dot{-}h)\,d\mu\right|$$

for $f \in V_n$. Combining these equations we obtain

$$||f||_2^2 = \frac{1}{p^n}\sum_{h\in H_0}\left|\int\limits_{\mathfrak{G}} f(x)\varphi(\mathcal{A}^{-n}x\dot{-}h)\,d\mu\right|^2 = \frac{1}{p^n}||f||_2^2,$$

for any $n \in \mathbb{N}$. It follows $f(x) = 0$ a.e. $\square$

**Theorem 3.4** *Let $\varphi$ be a solution of the equation (3.2) and $(\varphi(x\dot{-}h))_{h\in H_0}$ an orthonormal basis in $V_0$. Then $\overline{\bigcup\limits_{n\in\mathbb{Z}} V_n} = L_2(\mathfrak{G})$ if and only if*

$$\bigcup_{n\in\mathbb{Z}} \operatorname{supp}\hat{\varphi}(\cdot\mathcal{A}^{-n}) = X.$$

**Proof.** This theorem is written in [14] for any zero-dimensional group under the condition $|\hat{\varphi}| = \mathbf{1}_{\mathfrak{G}_0^\perp}$. But this condition was used to get the inclusion $V_n \subset V_{n+1}$ only. By theorems 3.2 the inclusion $V_n \subset V_{n+1}$ holds. Therefore the theorem is true. $\square$

The refinement equation (3.2) may be written in the form

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}), \tag{3.4}$$

where

$$m_0(\chi) = \frac{1}{p}\sum_{h\in H_0^{(N+1)}} \beta_h\overline{(\chi\mathcal{A}^{-1}, h)} \tag{3.5}$$

is a mask of the equation (3.4).

**Lemma 3.5** *Let $\varphi \in \mathfrak{D}_M(\mathfrak{G}_{-N})$. Then the mask $m_0(\chi)$ is constant on cosets $\mathfrak{G}_{-N}^{\perp}\zeta$.*

**Proof.** We will prove that $(\chi, \mathcal{A}^{-1}h)$ are constant on cosets $\mathfrak{G}_{-N}^{\perp}\zeta$. Without loss of generality, we can assume that $\zeta = r_{-N}^{\alpha_{-N}} \dots r_{-N+s}^{\alpha_{-N+s}} \notin \mathfrak{G}_{-N}^{\perp}$. If

$$h = a_{-1}g_{-1}\dot{+}\dots\dot{+}a_{-N-1}g_{-N-1} \in H_0^{(N+1)}$$

then

$$\mathcal{A}^{-1}h = a_{-1}g_0\dot{+}\dots\dot{+}a_{-N-1}g_{-N} \in \mathfrak{G}_{-N}.$$

If $\chi \in \mathfrak{G}_{-N}^{\perp}\zeta$ then $\chi = \chi_{-N}\zeta$ where $\chi_{-N} \in \mathfrak{G}_{-N}^{\perp}$. Therefore $(\chi, \mathcal{A}^{-1}h) = (\chi_{-N}\zeta, \mathcal{A}^{-1}h) = (\zeta, \mathcal{A}^{-1}h)$. This means that $(\chi, \mathcal{A}^{-1}h)$ depends on $\zeta$ only. $\square$

**Lemma 3.6** *The mask $m_0(\chi)$ is a periodic function with any period $r_1^{\alpha_1}r_2^{\alpha_2}\dots r_s^{\alpha_s}$ $(s \in \mathbb{N},\ \alpha_j = \overline{0, p-1},\ j = \overline{1, s})$.*

**Proof.** Using the equation $(r_k, g_l) = 1, (k \neq l)$ we find

$$(\chi r_1^{\alpha_1}r_2^{\alpha_2}\dots r_s^{\alpha_s}, \mathcal{A}^{-1}h) = (\chi r_1^{\alpha_1}r_2^{\alpha_2}\dots r_s^{\alpha_s}, a_{-1}g_0\dot{+}a_{-2}g_{-1}\dot{+}\dots\dot{+}a_{-N-1}g_{-N}) =$$

$$= (\chi, a_{-1}g_0\dot{+}a_{-2}g_{-1}\dot{+}\dots\dot{+}a_{-N-1}g_{-N}) = (\chi\mathcal{A}^{-1}, h).$$

Therefore $m_0(\chi r_1^{\alpha_1}\dots r_s^{\alpha_s}) = m_0(\chi)$ and the lemma is proved. $\square$

**Lemma 3.7** *The mask $m_0(\chi)$ is defined by its values on cosets $\mathfrak{G}_{-N}^{\perp}r_{-N}^{\alpha_{-N}}\dots r_0^{\alpha_0}$ $(\alpha_j = \overline{0, p-1})$.*

**Proof.** Let us denote

$$k = \alpha_0 + \alpha_{-1}p + \dots + \alpha_{-N}p^N \in [0, p^{N+1} - 1],$$

$$l = a_{-1} + a_{-2}p + \dots + a_{-N-1}p^N \in [0, p^{N+1} - 1].$$

Then (3.5) can be written as the system

$$m_0(\chi_k) = \frac{1}{p}\sum_{l=0}^{p^{N+1}-1}\beta_l\overline{(\chi_k, \mathcal{A}^{-1}h_l)},\ k = \overline{0, p^{N+1}-1} \tag{3.6}$$

in the unknowns $\beta_l$. We consider the characters $\chi_k$ on the subgroup $\mathfrak{G}_{-N_0}$. Since $\mathcal{A}^{-1}h_l$ lie in $\mathfrak{G}_{-N}$, it follows that the matrix $p^{-\frac{N+1}{2}}\overline{(\chi_k, \mathcal{A}^{-1}h_l)}$ is unitary, and so the system (3.6) has a unique solution for each finite sequence $(m_0(\chi_k))_{k=0}^{p^{N+1}-1}$.

**Remark.** The function $m_0(\chi)$ constructing in Lemma 3.7 may be not a mask for $\varphi \in \mathfrak{D}_M(\mathfrak{G}_{-N})$. In the section 4 we find conditions under which the function $m_0(\chi)$ will be a mask.

**Lemma 3.8** *Let $\hat{f}_0(\chi) \in \mathfrak{D}_{-N}(\mathfrak{G}_1^\perp)$. Then*

$$\hat{f}_0(\chi) = \frac{1}{p} \sum_{h \in H_0^{(N+1)}} \beta_h \overline{(\chi, \mathcal{A}^{-1}h)}. \tag{3.7}$$

**Prof.** Since $\int\limits_{\mathfrak{G}_0^\perp} (\chi, g)\overline{(\chi, h)}\, d\nu(\chi) = \delta_{h,g}$ for $h, g \in H_0$ it follows that

$\int\limits_{\mathfrak{G}_0^\perp} (\chi\mathcal{A}^{-1}, g)\overline{(\chi\mathcal{A}^{-1}, h)}\, d\nu(\chi) = p\delta_{h,g}$.

Therefore we can consider the set $\left(\frac{\mathcal{A}^{-1}h}{\sqrt{p}}\right)_{h \in H_0^{(N+1)}}$ as an orthonormal system on $\mathfrak{G}_1^\perp$. We know (lemma 3.5) that $(\chi, \mathcal{A}^{-1}h)$ is a constant on cosets $\mathfrak{G}_{-N}^\perp\zeta$. It is evident the dimensional of $\mathfrak{D}_{-N}(\mathfrak{G}_1^\perp)$ is equal to $p^{N+1}$. Therefore the system $\left(\frac{\mathcal{A}^{-1}h}{\sqrt{p}}\right)_{h \in H_0^{(N+1)}}$ is an orthonormal basis for $\mathfrak{D}_{-N}(\mathfrak{G}_1^\perp)$ and the equation (3.7) is valid. $\square$

# 4 The main results. The statements and proofs

In this section we find the necessary and sufficient condition under which a step function $\varphi(x) \in \mathfrak{D}_M(\mathfrak{G}_{-N})$ generates an orthogonal MRA on the $p$-adic Vilenkin group. We will prove also that for any $n \in \mathbb{N}$ there exists a step function $\varphi$ such that 1) $\varphi$ generate an orthogonal MRA, 2) $\operatorname{supp} \hat{\varphi} \subset \mathfrak{G}_n^\perp$, 3) $\hat{\varphi}(\mathfrak{G}_n^\perp \setminus \mathfrak{G}_{n-1}^\perp) \not\equiv 0$.

First we obtain a test under which the system of shifts $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system.

**Theorem 4.1** *Let $\varphi(x) \in \mathfrak{D}_M(\mathfrak{G}_{-N})$. A shift's system $(\varphi(x \dot{-} h))_{h \in H_0}$ will be orthonormal if and only if for any $\alpha_{-N}, \alpha_{-N+1}, \ldots, \alpha_{-1} = \overline{(0, p-1)}$*

$$\sum_{\alpha_0, \alpha_1, \ldots, \alpha_{M-1}=0}^{p-1} |\hat{\varphi}(\mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}} \ldots r_0^{\alpha_0} \ldots r_{M-1}^{\alpha_{M-1}})|^2 = 1. \tag{4.1}$$

**Proof.** First we prove that the system $(\varphi(x \dot{-} h))_{h \in H_0}$ will be orthonormal if and only if

$$\sum_{\alpha_{-N}, \ldots, \alpha_0, \ldots, \alpha_{M-1}} |\hat{\varphi}(\mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}} \ldots r_{M-1}^{\alpha_{M-1}})|^2 = p^N. \tag{4.2}$$

and for any vector $(a_{-1}, a_{-2}, \ldots, a_{-N}) \neq (0, 0, \ldots, 0)$, $(a_j = \overline{0, p-1})$

$$\sum_{\alpha_{-1}, \ldots, \alpha_{-N}} \exp\left(\frac{2\pi i}{p}(a_{-1}\alpha_{-1} + a_{-2}\alpha_{-2} + \cdots + a_{-N}\alpha_{-N})\right) \times$$

$$\times \sum_{\alpha_0,\alpha_1,\dots,\alpha_{M-1}} |\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_{M-1}^{\alpha_{M-1}})|^2 = 0 \qquad (4.3)$$

Let $(\varphi(x \dot{-} h))_{h \in H_0}$ be an orthonormal system. Using the Plansherel equality and Lemma 2.3 we have

$$\delta_{h_1 h_2} = \int\limits_{\mathfrak{G}} \varphi(x \dot{-} h_1) \overline{\varphi(x \dot{-} h_2)}\, d\mu(x) = \int\limits_{\mathfrak{G}_M^{\perp}} |\hat{\varphi}(\chi)|^2 (\chi, h_2 \dot{-} h_1)\, d\nu(\chi) =$$

$$= \sum_{\alpha_{-N},\dots,\alpha_0,\dots,\alpha_{M-1}} \int\limits_{\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}}} |\hat{\varphi}(\chi)|^2 (\chi, h_2 \dot{-} h_1)\, d\nu(\chi) =$$

$$= \sum_{\alpha_{-N},\dots,\alpha_{M-1}} |\hat{\varphi}(G_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}}|^2 \int\limits_{\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}}} (\chi, h_2 \dot{-} h_1)\, d\nu(\chi) =$$

$$= p^{-N} \mathbf{1}_{\mathfrak{G}_{-N}} (h_2 \dot{-} h_1) \times$$

$$\times \sum_{\alpha_{-N},\dots,\alpha_{M-1}} |\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}}|^2 (r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}}, h_2 \dot{-} h_1).$$

If $h_2 = h_1$, we obtain the equality (4). If $h_2 \neq h_1$ then

$$h_2 \dot{-} h_1 = a_{-1} g_{-1} \dot{+} \dots \dot{+} a_{-N} g_{-N} \in \mathfrak{G}_{-N} \qquad (4.4)$$

or

$$h_2 \dot{-} h_1 = a_{-1} g_{-1} \dot{+} \dots \dot{+} a_{-N} g_{-N} \dot{+} \dots \dot{+} a_{-s} g_{-s} \in \mathfrak{G} \setminus \mathfrak{G}_{-N}. \qquad (4.5)$$

If the condition (4.5) are fulfilled, then $\mathbf{1}_{\mathfrak{G}_{-N}}(h_2 \dot{-} h_1) = 0$. If the condition (4.4) are fulfilled, then

$$\mathbf{1}_{\mathfrak{G}_{-N}}(h_2 \dot{-} h_1) = 1,$$

$$(r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}}, h_2 \dot{-} h_1) = (r_{-N}, g_{-N})^{a_{-N}\alpha_{-N}} \dots (r_{-1}, g_{-1})^{a_{-1}\alpha_{-1}}.$$

Using the equality $(r_n, g_n) = e^{\frac{2\pi i}{p}}$ we obtain the equality (4).The conversely may be proved by analogy.

Let as show now if for any vector $(a_{-1}, a_{-2}, \dots, a_{-N}) \neq (0, 0, \dots, 0)$ the conditions (4.2) (4) are fulfilled, then for any $\alpha_{-N}, \alpha_{-N+1}, \dots, \alpha_{-1} = \overline{0, p-1}$

$$\sum_{\alpha_0,\alpha_1,\dots,\alpha_{M-1}} |\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}})|^2 = 1. \qquad (4.6)$$

Let us denote

$$n = \sum_{j=1}^{N} a_{-j} p^{j-1}, \quad k = \sum_{j=1}^{N} \alpha_{-j} p^{j-1}, \quad C_{n,k} = e^{\frac{2\pi i}{p} \left( \sum_{j=1}^{N} \alpha_{-j} a_{-j} \right)}.$$

and write the equalities (4.2) (4) as the system

$$\begin{array}{l} C_{0,0}x_0 + C_{0,1}x_1 + \cdots + C_{0,p^N-1}x_{p^N-1} = p^N \\ C_{1,0}x_0 + C_{1,1}x_1 + \cdots + C_{1,p^N-1}x_{p^N-1} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_{p^N-1,0}x_0 + C_{p^N-1,1}x_1 + \cdots + C_{p^N-1,p^N-1}x_{p^N-1} = 0 \end{array} \tag{4.7}$$

with unknowns

$$x_k = \sum_{\alpha_0,\alpha_1,\dots,\alpha_{M-1}} |\hat{\varphi}(G_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}})|^2.$$

The matrix $(C_{n,k})$ is orthogonal. Indeed, if
$(a_{-1}, a_{-2}, \dots, a_{-N}) \neq (a'_{-1}, a'_{-2}, \dots, a'_{-N})$, i.e., $k \neq n'$ we obtain

$$\sum_{k=0}^{p^N-1} C_{n,k}\overline{C_{n',k}} = \sum_{\alpha_{-1},\dots,\alpha_{-N}} \exp\left(\frac{2\pi i}{p}((a_{-1}-a'_{-1})\alpha_{-1} + (a_{-N}-a'_{-N})\alpha_{-N})\right) = 0,$$

so at least one of differences $a_{-l} - a'_{-l} \neq 0$. So, the system (4.7) has unique solution. It is evident that $x_k = 1$ is a solution of this system. This means that (4.6) is fulfil, and the necessity is proved. The sufficiency is evident. □

Now we obtain a necessary and sufficient conditions for function $m_0(\chi)$ to be a mask on the class $\mathfrak{D}_{-N}(\mathfrak{G}_M^{\perp})$, i.e. there exists $\hat{\varphi} \in \mathfrak{D}_{-N}(\mathfrak{G}_M^{\perp})$ for which

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}). \tag{4.8}$$

If $m_0(\chi)$ is a mask of (4.8) then
T1) $m_0(\chi)$ is constant on cosets $\mathfrak{G}_{-N}^{\perp}\zeta$,
T2) $m_0(\chi)$ is periodic with any period $r_1^{\alpha_1} r_2^{\alpha_2} \dots r_s^{\alpha_s}$, $\alpha_j = \overline{0, p-1}$,
T3) $m_0(\mathfrak{G}_{-N}^{\perp}) = 1$.
Therefore we will assume that $m_0$ satisfies these conditions. Let

$$E_k \subset \mathfrak{G}_k^{\perp} \setminus \mathfrak{G}_{k-1}^{\perp} \;\; ,(k = -N+1, -N+2, \dots, 0, 1, \dots, M, M+1)$$

be a set, on which $m_0(E_k) = 0$. Since $m_0(\chi)$ is constant on cosets $\mathfrak{G}_{-N}^{\perp}\zeta$, it follows that $E_k$ is a union of such cosets or $E_k = \emptyset$.

**Theorem 4.2** $m_0(\chi)$ *is a mask of some equation on the class* $\mathfrak{D}_{-N}(\mathfrak{G}_M^{\perp})$ *if and only if*

$$\bigcup_{k=-N+1}^{M+1} E_k \mathcal{A}^{M+1-k} = \mathfrak{G}_{M+1}^{\perp} \setminus \mathfrak{G}_M^{\perp}. \tag{4.9}$$

**Proof.** Since $m_0(\chi) = 1$ on $\mathfrak{G}_N$ it follows that $m_0(\chi\mathcal{A}^{-M-N}) = 1$ for $\chi \in \mathfrak{G}_M^\perp$. Therefore $m_0(\chi)$ will be a mask if and only if

$$m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N}) = 0 \tag{4.10}$$

on $\mathfrak{G}_{M+1}^\perp \setminus \mathfrak{G}_M^\perp$. Indeed, if (4.10) is true we set

$$\hat{\varphi}(\chi) = \prod_{k=0}^{\infty} m_0(\chi\mathcal{A}^{-k}) \in \mathfrak{D}_{-N}(\mathfrak{G}_M^\perp).$$

Then $\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$ and

$$m_0(\chi) = \sum_{h\in H_0^{(N+1)}} \beta_h\overline{(\chi\mathcal{A}^{-1}, h)}$$

for some $\beta_h$. Therefore $m_0(\chi)$ is a mask. Inversely let $m_0(\chi)$ be a mask, i.e. $\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}) \in \mathfrak{D}_{-N}(\mathfrak{G}_M^\perp)$. From it we find

$$\hat{\varphi}(\chi) = m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N})\hat{\varphi}(\chi\mathcal{A}^{-M-N-1}),$$

and $\hat{\varphi}(\chi\mathcal{A}^{-M-N-1}) = 1$ on $\mathfrak{G}_{M+1}^\perp$. Since $\hat{\varphi}(\chi) = 0$ on $\mathfrak{G}_{M+1}^\perp \setminus \mathfrak{G}_M^\perp$, it follows

$$m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N}) = 0$$

on $\mathfrak{G}_{M+1}^\perp \setminus \mathfrak{G}_M^\perp$.

To conclude the proof, it remains to note that for any $-N+1 \le k \le M+1$ the inclusion $E_k\mathcal{A}^{-k+M+1} \subset \mathfrak{G}_{M+1}^\perp \setminus \mathfrak{G}_M^\perp$ is true. Therefore the equation (4.9) is fulfil if and only if the equation (4.10) is true. $\square$

**Lemma 4.3** *Let $\hat{\varphi} \in \mathfrak{D}_{-N}(\mathfrak{G}_M^\perp)$ be a solution of the refinement equation*

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}).$$

*Then for any $\alpha_{-N}, \alpha_{-N+1}, \dots, \alpha_{-1} = \overline{0, p-1}$*

$$\sum_{\alpha_0=0}^{p-1} |m_0(\mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}} r_{-N+1}^{\alpha_{-N+1}} \dots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0})|^2 = 1. \tag{4.11}$$

**Proof.** Since $\hat{\varphi} \in \mathfrak{D}_{-N}(\mathfrak{G}_M^\perp)$, it follows that $\hat{\varphi}(\mathfrak{G}_{M+1}^\perp \setminus \mathfrak{G}_M^\perp) = 0$. Using theorem 4.1 we have

$$1 = \sum_{\alpha_0,\alpha_1,\dots,\alpha_{M-1}=0} |\hat{\varphi}(\mathfrak{G}_{-N}^\perp r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}})|^2 =$$

$$= \sum_{\alpha_0,\dots,\alpha_{M-1},\alpha_M=0} |\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0} \dots r_{M-1}^{\alpha_{M-1}} r_M^{\alpha_M})|^2 = \sum_{\alpha_0=0}^{p-1} |m_0(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0})|^2$$

$$\cdot \sum_{\alpha_1,\dots,\alpha_{M-1},\alpha_M=0} |\hat{\varphi}(\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}+1} \dots r_{-1}^{\alpha_0} r_0^{\alpha_1} \dots r_{M-2}^{\alpha_{M-1}} r_{M-1}^{\alpha_M})|^2 =$$

$$= \sum_{\alpha_0=0}^{p-1} |\mathfrak{G}_{-N}^{\perp} r_{-N}^{\alpha_{-N}} \dots r_0^{\alpha_0}|^2. \ \square$$

**Corollary.** *If $N = 1$ and $m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0}) = \lambda_{\alpha_{-1}+\alpha_0 p}$ then we can write the equations (4.11) in the form*

$$\sum_{\alpha_0=0}^{p-1} |\lambda_{\alpha_{-1}+\alpha_0 p}|^2 = 1. \qquad (4.12)$$

**Theorem 4.4** *Suppose the function $m_0(\chi)$ satisfies the conditions T1,T2,T3, (4.10), and the function*

$$\hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n})$$

*satisfies the condition (4.1). Then $\varphi \in \mathfrak{D}_M(\mathfrak{G}_{-N})$ generates an orthogonal MRA.*

**Proof.** It is evident that $\hat{\varphi} \in \mathfrak{D}_{-N}(\mathfrak{G}_M^{\perp})$, $\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$ and $(\varphi(x\dot{-}h))_{h\in H_0}$ is an orthonormal system. From theorems 3.4, 3.3, 3.2 we find that the function $\varphi$ generates an orthogonal MRA. $\square$

**Definition 4.1** *A mask $m_0(\chi)$ is called $N$-elementary ($N \in \mathbb{N}$) if it is constant on cosets $\mathfrak{G}_{-N}^{\perp}\chi$ and its modulus $|m_0(\chi)|$ take two values:0 and 1 only. The refinable function $\varphi$ with Fourier transform*

$$\hat{\varphi}(\chi) = \prod_{j=0}^{\infty} m_0(\chi \mathcal{A}^{-1})$$

*is called $N$-elementary too.*

**Theorem 4.5** *Let $m_0(\chi)$ be an 1-elementary mask such that*

$$\sum_{\alpha_0=0}^{p-1} |m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0})|^2 = 1$$

*for any* $\alpha_{-1} = \overline{0, p-1}$. *Let us denote*

$$E_0^{(0)} = \{\alpha = \overline{0, p-1} : \ m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha}) = 0\}$$

*and* $l = \sharp E_0^{(0)}$, $0 \le l \le p-2$. *If* $\hat{\varphi}(\chi) = \prod_{j=0}^{\infty} m_0(\chi \mathcal{A}^{-j})$, *then* $\hat{\varphi}(\mathfrak{G}_{l+1}^{\perp} \setminus \mathfrak{G}_l^{\perp}) = 0$.

**Proof.** Since

$$\mathfrak{G}_{l+1}^{\perp} \setminus \mathfrak{G}_l^{\perp} = \bigsqcup_{\alpha_l=1}^{p-1} \bigsqcup_{\alpha_{l-1},\dots,\alpha_{-1}=0}^{p-1} (\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_{l-1}^{\alpha_{l-1}} r_l^{\alpha_l})$$

we need prove that

$$\hat{\varphi}(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_{l-1}^{\alpha_{l-1}} r_l^{\alpha_l}) = 0$$

for $\alpha_l = \overline{1, p-1}$; $\alpha_{-1}, \dots, \alpha_{l-1} = \overline{0, p-1}$. Using a periodicity of $\varphi$ we can write

$$\hat{\varphi}(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_{l-1}^{\alpha_{l-1}} r_l^{\alpha_l}) =$$
$$m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_{l-1}^{\alpha_{l-1}} r_l^{\alpha_l}) \hat{\varphi}(\mathfrak{G}_{-2}^{\perp} r_{-2}^{\alpha_{-1}} r_{-1}^{\alpha_0} \dots r_{l-2}^{\alpha_{l-1}} r_{l-1}^{\alpha_l}) =$$
$$m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0}) \hat{\varphi}(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_0} r_0^{\alpha_1} \dots r_{l-2}^{\alpha_{l-1}} r_{l-1}^{\alpha_l}) = \dots =$$
$$m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0}) m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_0 \alpha_1}) \dots m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{l-1}} r_0^{\alpha_l}) m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_l}).$$

Let us denote $m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{k} r_0^{j}) = \lambda_{k+jp}$ and write $\hat{\varphi}$ in the form

$$\hat{\varphi}(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_{l-1}^{\alpha_{l-1}} r_l^{\alpha_l}) = \lambda_{\alpha_{-1}+\alpha_0 p} \cdot \lambda_{\alpha_0+\alpha_1 p} \dots \lambda_{\alpha_{l-1}+\alpha_l p} \cdot \lambda_{\alpha_l}.$$

We will consider numbers $\lambda_{k+jp}$ as elements of the matrix $\Lambda = (\lambda_{j,k})$, where $j$ is a number of a line, $k$ is a number of a column. Let us consider the product

$$\Pi = \lambda_{\alpha_{-1}+\alpha_0 p} \cdot \lambda_{\alpha_0+\alpha_1 p} \dots \lambda_{\alpha_{l-2}+\alpha_{l-1} p} \cdot \lambda_{\alpha_{l-1}+\alpha_l p} \cdot \lambda_{\alpha_l} \ \ (\alpha_l \ne 0).$$

We need prove that $\Pi = 0$ for $\alpha_j = \overline{0, p-1}$, $j = \overline{-1, l-1}$ and for $\alpha_l = \overline{1, p-1}$.

If $\alpha_l \in E_0^{(0)}$, then $\lambda_{\alpha_l} = 0$ and $\Pi = 0$. Let $\alpha_l \in E_0^{(1)}$ and $\alpha_l \ne 0$.

If $\lambda_{\alpha_{l-1}+\alpha_l p} = 0$, then $\Pi = 0$ and theorem is proved. Therefore we assume $|\lambda_{\alpha_{l-1}+\alpha_l p}| = 1$. In this case $\alpha_{l-1} \in E_0^{(0)}$ and $\alpha_{l-1} = 0$. Let us consider $\lambda_{\alpha_{l-2}+\alpha_{l-1} p}$. If $\lambda_{\alpha_{l-2}+\alpha_{l-1} p} = 0$ then $\Pi = 0$. Therefore we assume $|\lambda_{\alpha_{l-2}+\alpha_{l-1} p}| = 1$. In this case $\alpha_{l-1} \in E_0^{(0)}$ and $\alpha_{l-1} \ne \alpha_{l-1}$. Let us consider $\lambda_{\alpha_{l-3}+\alpha_{l-2} p}$. If $\lambda_{\alpha_{l-3}+\alpha_{l-2} p} = 0$ then $\Pi = 0$ and the theorem is proved. Therefore we assume $|\lambda_{\alpha_{l-3}+\alpha_{l-2} p}| = 1$. In this case $\alpha_{l-3} \in E_0^{(0)}$ and $\alpha_{l-3} \notin \{\alpha_{l-1}, \alpha_{l-2}\}$.

In the general case, if

$$|\lambda_{\alpha_{l-s}+\alpha_{l-s+1}p}|\cdot|\lambda_{\alpha_{l-s+1}+\alpha_{l-s+2}p}|\dots|\lambda_{\alpha_{l-1}+\alpha_l p}|\cdot|\lambda_{\alpha_l}|=1$$

and

$$\alpha_{l-s}\notin\{\alpha_{l-s+1},\alpha_{l-s+2},\dots,\alpha_{l-1}\},\ \ \alpha_{l-s}\in E_0^{(0)}$$

then we consider $\lambda_{\alpha_{l-s-1}+\alpha_{l-s}p}$. If $|\lambda_{\alpha_{l-s-1}+\alpha_{l-s}p}|=0$ then $\Pi=0$ and the theorem is proved. If $|\lambda_{\alpha_{l-s-1}+\alpha_{l-s}p}|=1$ then

$$\alpha_{l-s-1}\notin\{\alpha_{l-s},\alpha_{l-s+1},\dots,\alpha_{l-1}\},\ \ \alpha_{l-s-1}\in E_0^{(0)}.$$

We have two possible cases.
1) For some $s\le l$

$$\lambda_{\alpha_{l-s}+\alpha_{l-s+1}p}\cdot\lambda_{\alpha_{l-s+1}+\alpha_{l-s+2}p}\dots\lambda_{\alpha_{l-1}+\alpha_l p}\cdot\lambda_{\alpha_l}=0.$$

In this case $\Pi=0$, and the theorem is proved.
2) For $s=l$

$$|\lambda_{\alpha_0+\alpha_1p}|\cdot|\lambda_{\alpha_1+\alpha_2p}|\dots|\lambda_{\alpha_{l-1}+\alpha_l p}|\cdot|\lambda_{\alpha_l}|=1.$$

In this case $\lambda_{\alpha_{-1}+\alpha_0p}=0$ for $\alpha_{-l}=\overline{0,p-1}$, then $\Pi=0$ and the theorem is proved. $\square$

**Remark.** If $l=p-1$, then $m_0(\mathfrak{G}_0^\perp\setminus\mathfrak{G}_{-1}^\perp)\equiv 0$. It follow $\hat\varphi(\mathfrak{G}_0^\perp\setminus\mathfrak{G}_{-1}^\perp)$ and consequently $\operatorname{supp}\hat\varphi(\chi)=\mathfrak{G}_{-1}^\perp$. In this case the system of shift $(\varphi(x\dot-h))_{h\in H_0}$ is not orthonormal system.

If $l=0$, then $|m_0(\mathfrak{G}_0^\perp)|\equiv 1$ and the system of shifts $(\varphi(x\dot-h))_{h\in H_0}$ will be orthonormal if and only if $\hat\varphi(\mathfrak{G}_1^\perp\setminus\mathfrak{G}_0^\perp)\equiv 0$. In this case $\varphi$ generate an orthogonal MRA on any zero-dimensional group [14].

**Corollary.** *Let $\varphi\in\mathfrak{D}_M(\mathfrak{G}_{-N})$ be an 1-elementary refinable function and $\varphi$ generate an orthogonal MRA on p-adic Vilenkin group $\mathfrak{G}$ with $p\ge 3$. Then* $\operatorname{supp}\hat\varphi(\chi)\subset\mathfrak{G}_{p-2}^\perp$.

The next theorem shows the sharpness of this result.

**Theorem 4.6** *Let $\mathfrak{G}$ – be a p-adic Vilenkin group, $p\ge 3$. Then for any $1\le l\le p-2$ there exists an 1-elementary refinable function $\varphi\in\mathfrak{D}_l(\mathfrak{G}_{-1})$ that generate an orthogonal MRA on group $\mathfrak{G}$.*

**Proof.** We will find the Fourier transform $\hat\varphi$ as product

$$\hat\varphi(\chi)=\prod_{j=0}^{\infty}m_0(\chi\mathcal{A}^{-j}),$$

where the 1-elementary mask $m_0(\chi)$ is constant on cosets $\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{-1}}r_0^{\alpha_0}\dots r_s^{\alpha_s}$ ($s\in\mathbb{N}\bigsqcup\{0\}$). We will construct the mask $m_0(\chi)$ on the subgroup $\mathfrak{G}_1^\perp$ only,

since $m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \dots r_s^{\alpha_s}) = m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0})$. We will assume also that for any $\alpha_{-1} = \overline{0, p-1}$

$$\sum_{\alpha_0=0}^{p-1} |m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0})|^2 = 1, \tag{4.13}$$

since this condition is necessary for mask $m_0(\chi)$.

Choose an arbitrary set $E_l^{(0)} \subset \{1, 2, \dots, p-1\}$ of cardinality $\sharp E_l^{(0)} = l$. Let us denote $E_l^{(1)} = \{1, 2, \dots, p-1\} \setminus E_l^{(0)}$ and $m_0(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0}) = \lambda_{\alpha_{-1}+\alpha_0 p}$. First we set

$$\lambda_0 = 1, \quad |\lambda_\alpha| = \begin{cases} 0, & \alpha \in E_l^{(0)}, \\ 1, & \alpha \in E_l^{(1)} \end{cases}.$$

Now we will define $\lambda_{\alpha_{-1}+\alpha_0 p}$ for $\alpha_0 \geq 1$. It follow from (4.13) that $\lambda_{\alpha_{-1}+\alpha_0 p} = 0$ for $\alpha_{-1} \in E_l^{(1)}$, $\alpha_0 \geq 1$. Choose an arbitrary $\alpha_{l-1}^{(0)} \in E_l^{(1)}$ and fix it. Now we choose $\alpha_{l-2}^{(0)} \in E_l^{(0)}$ and set

$$|\lambda_{\alpha_{l-2}^{(0)}+\alpha_{l-1}^{(0)} p}| = 1, \quad |\lambda_{\alpha_{l-2}^{(0)}+\alpha p}| = 0 \text{ if } \alpha \neq \alpha_{l-1}^{(0)}.$$

If numbers $\alpha_{l-2}^{(0)}, \dots, \alpha_s^{(0)} \in E_l^{(0)}$ $(s = l-1, l-2, \dots, 0)$ have been choosen we choose $\alpha_{s-1}^{(0)} \in E_l^{(0)} \setminus \{\alpha_{l-2}^{(0)}, \dots, \alpha_s^{(0)}\}$ and set

$$|\lambda_{\alpha_{s-1}^{(0)}+\alpha_s^{(0)} p}| = 1, \quad |\lambda_{\alpha_{s-1}^{(0)}+\alpha p}| = 0 \text{ if } \alpha \neq \alpha_s^{(0)}.$$

So the mask $m_0(\chi)$ have been defined on the subgroup $\mathfrak{G}_1^{\perp}$ and consequently on the group $\mathfrak{G}$.

It is evident that

$$\lambda_{\alpha_{-1}^{(0)}+\alpha_0^{(0)} p} \cdot \lambda_{\alpha_0^{(0)}+\alpha_1^{(0)} p} \dots \lambda_{\alpha_{l-2}^{(0)}+\alpha_{l-1}^{(0)} p} \cdot \lambda_{\alpha_{l-1}^{(0)}} \neq 0.$$

Let us show that for any vector $(\alpha_{-1}, \alpha_0, \dots, \alpha_{l-1}) \neq (\alpha_{-1}^{(0)}, \alpha_0^{(0)}, \dots, \alpha_{l-1}^{(0)})$

$$\lambda_{\alpha_{-1}+\alpha_0 p} \cdot \lambda_{\alpha_0+\alpha_1 p} \dots \lambda_{\alpha_{l-2}+\alpha_{l-1} p} \cdot \lambda_{\alpha_{l-1}} = 0. \tag{4.14}$$

Indeed, if $\alpha_{l-1} \in E_l^{(0)}$ then $\lambda_{\alpha_{l-1}} = 0$. If $\alpha_{l-1} \in E_l^{(1)}$ and $\alpha_{l-1} \neq \alpha_{l-1}^{(0)}$ then $\lambda_{\alpha_{l-2}+\alpha_{l-1} p} = 0$. If $\alpha_{l-1} \in E_l^{(1)}$ and $\alpha_{l-1} = \alpha_{l-1}^{(0)}$ then we denote

$$s = \min\{j : \ \alpha_j = \alpha_j^{(0)}\}.$$

For this $s$ we have $\lambda_{\alpha_{s-1}+\alpha_s^{(0)} p} = 0$ and the equality (4.14) is proved. It should be noted that $\lambda_{\alpha+\alpha_{-1}^{(0)} p} = 0$ for $\alpha = \overline{0, p-1}$. Therefore

$$\lambda_{\alpha+\alpha_{-1}^{(0)} p} \cdot \lambda_{\alpha_{-1}^{(0)}+\alpha_0^{(0)} p} \dots \lambda_{\alpha_{l-2}^{(0)}+\alpha_{l-1}^{(0)} p} \cdot \lambda_{\alpha_{l-1}^{(0)}} = 0. \tag{4.15}$$

Let us show that $\hat\varphi(\mathfrak{G}_l^\perp\setminus\mathfrak{G}_{l-1}^\perp)\not\equiv 0$ and $\hat\varphi(\mathfrak{G}_{l+1}^\perp\setminus\mathfrak{G}_l^\perp)\equiv 0$. Since $m_0(\chi)$ is periodic with any period $r_1^{\alpha_1}r_2^{\alpha_2}\dots r_s^{\alpha_s}$, it follow that

$$\hat\varphi(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{-1}}r_0^{\alpha_0}\dots r_{l-1}^{\alpha_{l-1}})=$$

$$m_0(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{-1}}r_0^{\alpha_0}\dots r_{l-1}^{\alpha_{l-1}})m_0(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_0}r_0^{\alpha_1}\dots r_{l-1}^{\alpha_{l-1}})\dots m_0(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{l-2}}r_0^{\alpha_{l-1}})m_0(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{l-1}})=$$
$$=m_0(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{-1}}r_0^{\alpha_0})m_0(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_0}r_0^{\alpha_1})\dots m_0(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{l-2}}r_0^{\alpha_{l-1}})m_0(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{l-1}})=$$
$$=\lambda_{\alpha_{-1}+\alpha_0 p}\cdot\lambda_{\alpha_0+\alpha_1 p}\cdot\dots\cdot\lambda_{\alpha_{l-2}+\alpha_{l-1}p}\cdot\lambda_{\alpha_{l-1}}\neq 0$$

for $(\alpha_{-1},\alpha_0,\dots,\alpha_{l-2},\alpha_{l-1})=(\alpha_{-1}^{(0)},\alpha_0^{(0)},\dots,\alpha_{l-2}^{(0)},\alpha_{l-1}^{(0)})$. This means that $\hat\varphi(\mathfrak{G}_l^\perp\setminus\mathfrak{G}_{l-1}^\perp)\not\equiv 0$.

By analogy

$$\hat\varphi(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{-1}}r_0^{\alpha_0}\dots r_{l-1}^{\alpha_{l-1}}r_l^{\alpha_l})=\lambda_{\alpha_{-1}+\alpha_0 p}\cdot\lambda_{\alpha_0+\alpha_1 p}\cdot\dots\cdot\lambda_{\alpha_{l-1}+\alpha_l p}\cdot\lambda_{\alpha_l}.$$

If $\alpha_l\in E_l^{(0)}$ then $\lambda_{\alpha_l}=0$. If $\alpha_l\in E_l^{(1)}$ and $\alpha_l\neq\alpha_{l-1}^{(0)}$ then $\lambda_{\alpha_{l-1}+\alpha_l p}=0$ for any $\alpha_{l-1}=\overline{0,p-1}$. If $\alpha_l\in E_l^{(1)}$ and $\alpha_l=\alpha_{l-1}^{(0)}$ we define the number

$$s=\min\{j:\ \alpha_j=\alpha_{j-1}^{(0)}\}.$$

Then

$$\hat\varphi(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{-1}}r_0^{\alpha_0}\dots r_l^{\alpha_l})=\lambda_{\alpha_{-1}+\alpha_0 p}\dots\lambda_{\alpha_{s-1}+\alpha_{s-1}^{(0)}p}\lambda_{\alpha_{s-1}^{(0)}+\alpha_{s-2}^{(0)}p}\dots\lambda_{\alpha_{l-2}^{(0)}+\alpha_{l-1}^{(0)}p}\cdot\lambda_{\alpha_{l-1}^{(0)}}=0$$

since $\lambda_{\alpha_{s-1}+\alpha_{s-1}^{(0)}p}=0$ for any $\alpha_{s-1}=\overline{0,p-1}$. This means that $\hat\varphi(\mathfrak{G}_{l+1}^\perp\setminus\mathfrak{G}_l^\perp)\equiv 0$. Consequently $\hat\varphi\in\mathfrak{D}_{-1}(\mathfrak{G}_l^\perp)$.

Let us show that $(\varphi(x\dot-h))_{h\in H_0}$ is an orthonormal system. We need show that the sum

$$S(\alpha_{-1})=\sum_{\alpha_0,\alpha_1,\dots,\alpha_{l-1}=0}^{p-1}|\hat\varphi(\mathfrak{G}_{-1}^\perp r_{-1}^{\alpha_{-1}}r_0^{\alpha_0}\dots r_{l-1}^{\alpha_{l-1}})|^2=$$

$$=\sum_{\alpha_0,\alpha_1,\dots,\alpha_{l-1}=0}^{p-1}|\lambda_{\alpha_{-1}+\alpha_0 p}|^2|\lambda_{\alpha_0+\alpha_1 p}|^2\dots|\lambda_{\alpha_{l-2}+\alpha_{l-1}p}|^2|\lambda_{\alpha_{l-1}}|^2=1$$

for any $\alpha_{-1}=\overline{0,p-1}$.

Let us consider next possible cases.

1) If $\alpha_{-1}=0$ then $\lambda_{\alpha_{-1}+\alpha_0 p}\neq 0$ iff $\alpha_0=0$, $\lambda_{\alpha_0+\alpha_1 p}\neq 0$ iff $\alpha_1=0$ and so on.

Consequently $S(\alpha_{-1})\neq 0$ iff $\alpha_{-1}=\alpha_0=\dots=\alpha_{l-1}=0$. It means that $S(\alpha_{-1})=1$.

2) If $\alpha_{-1}\neq 0$ and $\alpha_{-1}\in E_l^{(1)}$ then $\lambda_{\alpha_{-1}+\alpha_0 p}\neq 0$ iff $\alpha_0=0$ and by analog

$S(\alpha_{-1}) = 1$.
3) If $\alpha_{-1} \in E_l^{(0)}$ and $\alpha_{-1} = \alpha_{-1}^{(0)}$ then $\lambda_{\alpha_{-1}+\alpha_0 p} \neq 0$ iff $\alpha_0 = \alpha_0^{(0)}$, $\lambda_{\alpha_0^{(0)}+\alpha_1 p} \neq 0$ iff $\alpha_1 = \alpha_1^{(0)}$ and so on. Consequently $S(\alpha_{-1}) \neq 0$ iff $\alpha_0 = \alpha_0^{(0)}$, $\alpha_1 = \alpha_1^{(0)}, \ldots, \alpha_{l-1} = \alpha_{l-1}^{(0)}$. It means that $S(\alpha_{-1}) = 1$.
4) If $\alpha_{-1} \in E_l^{(0)}$ and $\alpha_{-1} = \alpha_j^{(0)}$ $(j \geq 0)$ then $\lambda_{\alpha_{-1}+\alpha_0 p} \neq 0$ iff $\alpha_0 = \alpha_{j+1}^{(0)}$, $\lambda_{\alpha_0+\alpha_1 p} \neq 0$ iff $\alpha_1 = \alpha_{j+2}^{(0)}$ and so on, $\alpha_{l-j-2} = \alpha_{l-1}^{(0)}$. Then $\alpha_{l-j-1} = \cdots = \alpha_{l-1} = 0$. This means that $S(\alpha_{-1}) = 1$. $\square$

By theorem 4.4 $\varphi(x)$ generate an orthogonal MRA. $\square$

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